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Calculation of the Undulator Radiation Spectra

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The frequency spectral and angular distribution of undulator radiation has been calculated for the case of a spatially periodic sinusoidal magnetic field. The results are expressed in terms of an integral form and also in a series of Bessel functions.

1. Introduction

The properties of synchrotron radiations from undulators have been investigated by several authors [1-4]. Different authors have different expressions for the frequency spectral and angular distribution of the radiation. The purpose of this report is to clarify different notations and expressions of the radiation by deriving the spectra in detail.

Some notations used in this report are following:

λ_u = length of the undulator period,

N = number of the periods,

βc = electron speed,

$\gamma = (1 - \beta^2)^{-1/2}$,

$\beta^* c$ = average electron velocity in the Z-direction,

$K = \dfrac{eB\lambda_u}{2\pi m_o c^2 \beta^*}$ undulator deflection parameter,

$\Psi_o = \dfrac{K}{\gamma}$,

$\omega_o = \dfrac{2\pi\beta^* c}{\lambda_u}$.

2. Electron Motion in a Transverse Undulator

The coordinate system of Fig. 1 is used in this report in order to relate the angular dependence of the radiated photon intensity with the rms beam size and divergence of an electron beam. Let us consider that relativistic electrons propagate along the Z-direction, the middle of an undulator of length $\lambda_u N$ is located at the origin of the coordinate system, and the undulator radiation is observed at point P located at a great distance R (R >> λ_uN).

For a spatially sinusoidal magnetic field of the undulator

$$\vec{B}_u = (B_x, B_y, B_z) = (0, B \sin \frac{2\pi z}{\lambda_u}, 0),$$

the equation of the motion of an electron

$$\gamma m_o \dot{\vec{\beta}} = \frac{e}{c} \vec{\beta} \times \vec{B}_u$$

can be decomposed as

$$\ddot{x} = -\frac{eB}{\gamma m_o c} \dot{z} \sin\left(\frac{2\pi z}{\lambda_u}\right),$$

$$\ddot{y} = 0,$$

$$\ddot{z} = \frac{eB}{\gamma m_o c} \dot{x} \sin\left(\frac{2\pi z}{\lambda_u}\right). \qquad (1)$$

For a highly relativistic electron with small amplitude in the X-direction (ψ_o << 1), the solutions of $\vec{\beta}$ and $\vec{r}$ may be expressed as

$$\vec{\beta} = \begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix} = \begin{pmatrix} x_0'/c + \beta\psi_0 \cos\omega_0 t \\ y_0'/c \\ \beta^* - \frac{x_0'}{c}\psi_0 \cos\omega_0 t - \frac{1}{4}\beta\psi_0^2 \cos 2\omega_0 t \end{pmatrix}, \tag{2}$$

$$\vec{r} = \begin{pmatrix} x_0' t + \frac{\beta c \psi_0}{\omega_0} \sin\omega_0 t \\ y_0' t \\ \beta^* c t - \frac{x_0'\psi_0}{\omega_0} \sin\omega_0 t - \frac{\beta c \psi_0^2}{8\omega_0} \sin 2\omega_0 t \end{pmatrix}, \tag{3}$$

where the x_0'/c and y_0'/c are initial velocities of the electron in the X- and Y-directions, respectively. In the Z-direction, the average β^* may be approximated as

$$\beta^* = \beta\left[1 - \frac{1}{4}\psi_0^2 - \frac{x_0'^2 + y_0'^2}{2\beta^2 c^2}\right]. \tag{4}$$

3. Calculation of the Undulator Spectra

The energy radiated per unit bandwidth by an electron into a unit solid angle is given by

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \vec{n}\times(\vec{n}\times\vec{\beta})\, e^{i\omega(t - \vec{n}\cdot\vec{r}/c)}\, dt\right|^2. \tag{5}$$

For small angles of θ and ϕ, and $R \gg r$, the unit vectors of $\vec{n}$ and $\vec{n}_0$ in Fig. 1 may be approximated as

$$\vec{n} \cong \vec{n}_0 = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\phi \\ \cos\theta\cos\phi \end{pmatrix} \cong \begin{pmatrix} \theta \\ \phi \\ 1 - \frac{1}{2}(\theta^2 + \phi^2) \end{pmatrix}. \tag{6}$$

Using Eqs. (3) and (6), the exponential factor in Eq. (5) becomes

$$e^{i\omega(t - \vec{n}\cdot\vec{r}/c)} = e^{i(\nu\omega_0 t - \rho\sin\omega_0 t + q\sin 2\omega_0 t)} \tag{7}$$

with

$$\begin{aligned} \nu &= \left(1 - \frac{x_0'}{c}\theta - \frac{y_0'}{c}\phi - \beta^* + \beta^*\frac{\theta^2+\phi^2}{2}\right)\frac{\omega}{\omega_0}, \\ \rho &= \left(\beta\psi_0\theta - \frac{x_0'}{c}\psi_0 - \frac{x_0'}{c}\psi_0\,\frac{\theta^2+\phi^2}{2}\right)\frac{\omega}{\omega_0}, \\ q &= \frac{\beta\psi_0}{8}\left(1 - \frac{\theta^2+\phi^2}{2}\right)\frac{\omega}{\omega_0}. \end{aligned} \tag{8}$$

To the order of γ^{-2}, Eq. (8) can be further simplified,

$$\begin{aligned} \nu &= \left(\frac{1}{2\gamma^2} + \frac{1}{4}\beta\psi_0^2 + \beta\gamma^2\frac{\Delta\theta^2+\Delta\phi^2}{2\gamma^2}\right)\frac{\omega}{\omega_0}, \\ \rho &= \beta\psi_0\,\frac{\gamma\Delta\theta}{\gamma}\,\frac{\omega}{\omega_0}, \\ q &= \frac{1}{8}\beta\psi_0^2\frac{\omega}{\omega_0}, \end{aligned} \tag{9}$$

where $\Delta\theta$ and $\Delta\phi$ are the angles between the direction of the observation and the initial angle of the electron in the beam,

$$\begin{aligned} \Delta\theta &= \theta - x_0'/c \\ \Delta\phi &= \phi - y_0'/c. \end{aligned} \tag{10}$$

Using Eqs. (2) and (6), the approximation of $\vec{n} \times (\vec{n} \times \vec{\beta})$ to the order of γ^{-1} is

$$\vec{n}_o \times (\vec{n}_o \times \vec{\beta}) = \begin{pmatrix} \frac{\gamma\Delta\theta}{\gamma} - \beta\psi_o \cos\omega_o t \\ \frac{\gamma\Delta\phi}{\gamma} \\ 0 \end{pmatrix}. \tag{11}$$

Substituting Eqs. (7) and (12) into Eq. (5), one obtains

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left(G_x^{\,2} + G_y^{\,2}\right), \tag{12}$$

where

$$G_x = \int_{-\infty}^{\infty}\left(\frac{\gamma\Delta\theta}{\gamma} - \beta\psi_o \cos\omega_o t\right)\cdot e^{i(\nu\omega_o t - p\sin\omega_o t + q\sin 2\omega_o t)}\,dt,$$

$$G_y = \int_{-\infty}^{\infty}\frac{\gamma\Delta\phi}{\gamma}\,e^{i(\nu\omega_o t - p\sin\omega_o t + q\sin 2\omega_o t)}\,dt.$$

In the moving frame along the Z-direction with velocity $\beta^* c$, electron oscillates with an angular frequency $\gamma^*\omega_o$. In the laboratory frame $\gamma^*\omega_o$ is Doppler-shifted to have the fundamental harmonic ω_1,

$$\omega_1 = \frac{\gamma^*\omega_o}{\gamma^*(1-\beta^*\cos\theta\cos\phi)}$$

$$= \frac{2\gamma^2\omega_o}{1+\frac{K^2}{2}+\gamma^2(\Delta\theta^2+\Delta\phi^2)}. \tag{13}$$

Then, Eq. (9) becomes

$$\nu = \frac{\omega}{\omega_1},$$

$$\rho = \frac{2\nu K \gamma\Delta\theta}{1+\frac{K^2}{2}+\gamma^2(\Delta\theta^2+\Delta\phi^2)}, \tag{14}$$

$$q = \frac{\nu K^2/4}{1+\frac{K^2}{2}+\gamma^2(\Delta\theta^2+\Delta\phi^2)}.$$

Frequency spectral and angular distribution of Eq. (12) can be exactly expressed in terms of an integral form (see Appendix A for details)

$$\frac{dI(\omega)}{d\Omega} = \frac{4e^2\gamma^2N^2}{c} W(\nu,N)\left(\frac{\nu}{1+\frac{K^2}{2}+\gamma^2(\Delta\theta^2+\Delta\phi^2)}\right)^2(g_x^2+g_y^2), \tag{15}$$

with

$$W(\nu,N) = \left(\frac{\sin N\pi\nu}{N\sin\pi\nu}\right)^2,$$

$$g_x = \gamma\Delta\theta\left[L_\nu(\rho,q) - \frac{\sin\pi\nu}{\pi\nu}\right] + \frac{K}{2}\left[L_{\nu+1}(\rho,q) + L_{\nu-1}(\rho,q)\right],$$

$$g_y = \gamma\Delta\phi\left[L_\nu(\rho,q) - \frac{\sin\pi\nu}{\pi\nu}\right],$$

$$L_\nu(p,q) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\nu\alpha + p\sin\alpha + q\sin 2\alpha)}\, d\alpha . \tag{16}$$

Here the effects of the finite length of the undulator on the distribution of the radiation is taken into account by the function $W(\nu,N)$. The functions g_x and g_y are independent of N. The factor $W(\nu,N)$ has principal maxima when ν is positive integers. Between two adjacent principal maxima $W(\nu,N)$ vanishes (N-1) times at $\nu = m/N$. It should be noted that ν is defined as the ratio of ω/ω_1. As shown in Eq. (13), the fundamental harmonic ω_1 is a function of the observation angles. Here g_x and g_y are the components of the polarization in the X- and Y-directions, respectively. When the observation point P is not on the Z-axis, the polarization angle depends on $\Delta\theta$ and $\Delta\phi$.

Equation (12) can also be expressed in terms of Bessel functions (see Appendix B for details):

$$\frac{dI(\omega)}{d\Omega} = \frac{4e^2\gamma^2N^2}{c}\sum_{k=1}^{\infty} F(\nu,k)\left(\frac{\nu}{1+\frac{K^2}{2}+\gamma^2(\Delta\theta^2+\Delta\phi^2)}\right)^2 \cdot \left(f_x^2 + f_y^2\right),$$

where

$$f_x = \gamma\Delta\theta\, S_1 - \frac{1+\frac{K^2}{2}+\gamma^2(\Delta\theta^2+\Delta\phi^2)}{2\gamma\Delta\theta}\left(\frac{k}{\nu}S_1 + \frac{2}{\nu}S_2\right),$$

$$f_y = \gamma\Delta\phi\, S_1 , \tag{17}$$

$$S_1 \equiv \sum_{n=-\infty}^{\infty} J_{k+2n}(p)\, J_n(q),$$

$$S_2 \equiv \sum_{n=-\infty}^{\infty} n\, J_{k+2n}(p)\, J_n(q).$$

In the limit of $N \to \infty$,

$$F(\nu, k) \to \frac{\omega_1}{N}\, \delta(\omega - k\omega_1).$$

4. Special Case

In the forward direction ($\Delta\theta = 0$, $\Delta\phi = 0$, $\theta = 0$, $\phi = 0$),

$$\omega_1 = \frac{2\gamma^2\omega_0}{1 + K^2/2},$$

$$\nu = \omega/\omega_1,$$

$$p = 0,$$

$$q = \frac{\nu K^2/4}{1 + K^2/2}. \tag{18}$$

$$g_x = \frac{K}{2}\left[L_{\nu+1}(0, q) + L_{\nu-1}(0, q)\right],$$

$$g_y = 0. \tag{19}$$

Using the Fourier expansion of Bessel functions of Eq. (B1), the integral of Eq. (19) becomes

$$L_{\nu\pm1}(0, q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i[(\nu\pm1)\alpha + q\sin 2\alpha]}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\nu\pm1)\alpha}\left\{\sum_{n=-\infty}^{\infty} e^{i2n\alpha} J_n(g)\right\} d\alpha .$$

To have non-vanishing values of the integral

$$\nu \pm 1 + 2n = 0$$

or

$$n = -\frac{\nu\pm1}{2} .$$

$$g_x = \frac{K}{2}\left\{(-1)^{\frac{\nu+1}{2}} J_{\frac{\nu+1}{2}}(g) + (-1)^{\frac{\nu-1}{2}} J_{\frac{\nu-1}{2}}(g)\right\}$$

$$= \frac{K}{2}\sin\frac{\pi\nu}{2}\left\{J_{\frac{\nu-1}{2}}(g) - J_{\frac{\nu+1}{2}}(g)\right\}. \qquad (20)$$

$$\frac{dI(\omega)}{d\Omega}_{\theta=\phi=0} = \frac{e^2\gamma^2N^2}{c}\left(\frac{\sin N\pi\nu}{2N\cos\frac{\pi\nu}{2}}\right)^2 F_\nu(K), \qquad (21)$$

where

$$F_\nu(K) = \left(\frac{\nu K}{1+\frac{K^2}{2}}\right)^2\left[J_{\frac{\nu-1}{2}}(g) - J_{\frac{\nu+1}{2}}(g)\right]^2. \qquad (22)$$

Equation (21) vanishes at even integers of ν.

For an electron beam of current I and bandwidth $\Delta\omega$ of the spectra, numerical values of Eq. (21) in terms of the number of photons is given by

$$\frac{dn(\omega)}{d\Omega} = \gamma^2N^2I\,\frac{\Delta\omega}{\omega}\,\frac{\alpha}{e}\left(\frac{\sin N\pi\nu}{2N\cos\frac{\pi\nu}{2}}\right)^2 F_\nu(K), \qquad (23)$$

where $$\alpha = \frac{e^2}{\hbar^2 c} = \frac{1}{137},$$

$$e = 1.6 \times 10^{-19} \text{ coul.}$$

The total photon flux $f_\nu(\omega)$ over the diffraction solid angle

$$2\pi \sigma_R'^2 = 2\pi \frac{1+\frac{K^2}{2}}{2\nu\gamma^2 N} \tag{24}$$

is

$$f_\nu(\omega) = 2\pi\sigma_R'^2 \left.\frac{dn(\omega)}{d\Omega}\right)_{\theta=\phi=0}$$
$$= (4.56\times 10^{16})\pi N I \frac{\Delta\omega}{\omega}\left(\frac{\sin N\pi\nu}{2N\cos\frac{\pi\nu}{2}}\right)^2 \frac{1+\frac{K^2}{2}}{\nu} F_\nu(K). \tag{25}$$

5. Expression in the Spherical Coordinate System

Derived formulae in the coordinate system of Fig. 1 can be expressed in the spherical coordinate system of Fig. 2. Here x_0' and y_0' are assumed to be zero.

(4)→ $$\beta^* = \beta\left(1-\frac{1}{4}\psi_0^2\right).$$

(6)→ $$\vec{n} \simeq \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{26}$$

$$(11)\rightarrow \quad \vec{n} \times (\vec{n} \times \vec{\beta}) = \begin{pmatrix} \beta^* \sin\theta \cos\phi - \beta\psi_0 \cos\omega_0 t \\ \beta^* \sin\theta \sin\phi \\ 0 \end{pmatrix}. \tag{27}$$

$$(13)\rightarrow \quad \omega_1 = \frac{\gamma^* \omega_0}{\gamma^* (1 - \beta^* \cos\theta)}$$

$$= \frac{2\gamma^2 \omega_0}{1 + \frac{K^2}{2} + \gamma^2\theta^2}. \tag{28}$$

$$\nu = \omega / \omega_1 ,$$

$$p = \frac{2\nu K (\gamma\theta) \cos\phi}{1 + \frac{K^2}{2} + \gamma^2\theta^2} ,$$

$$q = \frac{\nu K^2/4}{1 + \frac{K^2}{2} + \gamma^2\theta^2} . \tag{29}$$

$$\frac{dI(\omega)}{d\Omega} = \frac{4e^2\gamma^2 N^2}{c} W(\nu, N) \left(\frac{\nu}{1 + \frac{K^2}{2} + \gamma^2\theta^2} \right)^2 (g_x^2 + g_y^2),$$

$$W(\nu, N) = \left(\frac{\sin N\pi\nu}{N \sin\pi\nu} \right)^2, \tag{30}$$

$$g_x = (\gamma\theta)\cos\phi\left[L_\nu(p,q) - \frac{\sin\pi\nu}{\pi\nu}\right] + \frac{K}{2}\left[L_{\nu+1}(p,q) + L_{\nu-1}(p,q)\right],$$

$$g_y = (\gamma\theta)\sin\phi\left[L_\nu(p,q) - \frac{\sin\pi\nu}{\pi\nu}\right],$$

$$L_\nu(p,q) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\nu\alpha + p\sin\alpha + q\sin 2\alpha)}\,d\alpha.$$

Appendix A. Derivation of Eq. (15)

The integral of the first term of G_x in Eq. (13) in the region of the undulator is

$$\int_{-N\pi/\omega_0}^{N\pi/\omega_0} \frac{\gamma\Delta\theta}{\gamma} e^{i(\nu\omega_0 t - p\sin\omega_0 t + q\sin 2\omega_0 t)}\,d\alpha$$

$$= \frac{\gamma\Delta\theta}{\gamma}\frac{1}{\omega_0}\int_0^{2N\pi} e^{i[\nu\alpha - (-1)^N p\sin\alpha + q\sin 2\alpha]}\,d\alpha, \qquad \text{(A1)}$$

where $\omega_0 t = N\pi+\alpha$ is substituted. The sign of $p\sin\alpha$ depends on the number of the undulator periods N. To avoid this, one end of the undulator will be assumed to be at the origin of the coordinate system. Then,

$$\frac{\gamma\Delta\theta}{\gamma}\frac{1}{\omega_0}\int_0^{2N\pi} e^{i(\nu\alpha - p\sin\alpha + q\sin 2\alpha)}\,d\alpha$$

$$= \frac{\gamma\Delta\theta}{\gamma}\frac{1}{\omega_0}\left[\int_0^{2\pi} + \int_{2\pi}^{4\pi} + \cdots + \int_{2(N-1)\pi}^{2N\pi}\right] \cdot e^{i(\nu\alpha - p\sin\alpha + q\sin 2\alpha)} d\alpha . \tag{A2}$$

By substituting $\alpha = \theta$, $\alpha = 2\pi+\theta$, $\cdots\cdot$ $\alpha = 2(N-1)\pi+\theta$, for each integral,

$$(A2) = \frac{\gamma\Delta\theta}{\gamma}\frac{1}{\omega_0}\left[1 + e^{i2\pi\nu} + \cdots + e^{i(N-1)2\pi\nu}\right] \cdot \int_0^{2\pi} e^{i(\nu\theta - p\sin\theta + q\sin 2\theta)} d\theta$$

$$= \frac{\gamma\Delta\theta}{\gamma}\frac{2\pi N}{\omega_0} e^{iN\pi\nu} \frac{\sin N\pi\nu}{N\sin\pi\nu} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\nu\alpha + p\sin\alpha + q\sin 2\alpha)} d\alpha . \tag{A3}$$

Outside of the undulator where K = 0 (p=q=0),

$$\left[\int_{-\infty}^{0} + \int_{2N\pi}^{\infty}\right] e^{i\nu\alpha} d\alpha = -\int_0^{2N\pi} e^{i\nu\alpha} d\alpha$$

$$= -2\pi e^{iN\pi\nu} \frac{\sin N\pi\nu}{\pi\nu} . \tag{A4}$$

Then, the first term of G_x becomes

$$\frac{\gamma\Delta\theta}{\gamma}\frac{2\pi N}{\omega_0} e^{iN\pi\nu}\left(\frac{\sin N\pi\nu}{N\sin\pi\nu}\right)\left[L_\nu(p,q) - \frac{\sin\pi\nu}{\pi\nu}\right] . \tag{A5}$$

The integral of the second term of G_x is

$$-\beta\psi_0 \int_0^{2N\pi/\omega_0} \cos\omega_0 t \; e^{i(\nu\omega_0 t - p\sin\omega_0 t + q\sin 2\omega_0 t)} dt$$

$$= \frac{-\beta\psi_0}{2\omega_0} \int_0^{2N\pi} \Big[e^{i(\nu\alpha + \alpha - p\sin\alpha + q\sin 2\alpha)} + e^{i(\nu\alpha - \alpha - p\sin\alpha + q\sin 2\alpha)} \Big] d\alpha$$

$$= \frac{-\beta\psi_0}{2} \frac{2\pi N}{\omega_0} \Big\{ e^{i\pi N(\nu+1)} \frac{\sin N\pi(\nu+1)}{N\sin\pi(\nu+1)} L_{\nu+1}(p,q) + e^{i\pi N(\nu-1)} \frac{\sin N\pi(\nu-1)}{N\sin\pi(\nu-1)} L_{\nu-1}(p,q) \Big]$$

$$= \frac{K}{2\gamma} \frac{2\pi N}{\omega_0} e^{i\pi N\nu} \left(\frac{\sin N\pi\nu}{N\sin\pi\nu}\right) \Big[L_{\nu+1}(p,q) + L_{\nu-1}(p,q) \Big]. \qquad \text{(A6)}$$

The phase factor $e^{iN\pi\nu}$ in Eqs. (A5) and (A6) is due to the change of the undulator location in the coordinate system. The calculation of g_y is the same as that of the first term of g_x except the angular factor of $\Delta\phi$ instead of $\Delta\theta$.

Appendix B. Derivation of Eq. (17)

Using the Fourier expansions of Bessel functions

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} e^{in\theta} J_n(x),$$

$$e^{-ix\sin\theta} = \sum_{n=-\infty}^{\infty} (-1)^n e^{in\theta} J_n(x), \tag{B1}$$

and

$$J_{-n}(x) = (-1)^n J_n(x),$$

$$J_n(-x) = (-1)^n J_n(x),$$

the first term of G_x in Eq. (13) becomes

$$\frac{\gamma\Delta\theta}{\gamma}\int_{-N\pi/\omega_0}^{N\pi/\omega_0} e^{i(\nu\omega_0 t - p\sin\omega_0 t + q\sin 2\omega_0 t)}dt$$

$$= \frac{\gamma\Delta\theta}{\gamma}\frac{N}{\omega_0}\int_{-\pi}^{\pi} e^{i\nu N\theta}\left\{\sum_{n=-\infty}^{\infty} e^{-inN\theta} J_n(p)\right\}\cdot$$

$$\cdot\left\{\sum_{m=-\infty}^{\infty} e^{i2mN\theta} J_m(q)\right] d\theta$$

$$= \frac{\gamma\Delta\theta}{\gamma}\frac{2\pi N}{\omega_0}\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty} J_{k+2m}(p)\, J_m(q)\,\cdot$$

$$\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iN(\nu-k)\theta} d\theta. \tag{B2}$$

In the above derivations $\omega_o t = N\theta$ and $n-2m = k$ are substituted.

After calculating the integral Eq. (B2) becomes

$$\frac{\gamma\Delta\theta}{\gamma}\ \frac{2\pi N}{\omega_0} \sum_{k=1}^{\infty} \frac{\sin N\pi(\nu-k)}{N\pi(\nu-k)}\, S_1 \tag{B3}$$

where

$$S_1 \equiv \sum_{m=-\infty}^{\infty} J_{k+2m}(p)\, J_m(q).$$

The second term of G_x is

$$-\beta\psi_0 \int_{-N\pi/\omega_0}^{N\pi/\omega_0} \tfrac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{i(\nu\omega_0 t - p\sin\omega_0 t + q\sin 2\omega_0 t)}\, dt$$

$$= \frac{-\beta\psi_0}{2}\frac{N}{\omega_0}\int_{-\pi}^{\pi}\left[e^{i(\nu+1)N\theta} + e^{i(\nu-1)N\theta}\right]\left\{\sum_{n=-\infty}^{\infty} e^{-inN\theta} J_n(p)\right\}\cdot$$

$$\left\{\sum_{m=-\infty}^{\infty} e^{i2mN\theta} J_m(q)\right\} d\theta$$

$$= -\frac{\beta\psi_0}{2}\ \frac{2\pi N}{\omega_0}\sum_{k=1}^{\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\nu-k)N\theta} d\theta\right)\sum_{m=-\infty}^{\infty}\Big\{$$

$$J_{k+2m+1}(p)\, J_m(q) + J_{k+2m-1}(p)\, J_m(q)\Big\}. \tag{B4}$$

Calculating the integral in Eq. (B3) and using a recursion relation for $J_n(x)$, namely

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \tag{B5}$$

$$(B4) = -\frac{\beta\Psi_0}{2}\frac{2\pi N}{\omega_0}\sum_{k=1}^{\infty}\frac{\sin N\pi(\nu-k)}{N\pi(\nu-k)}\cdot$$

$$\cdot\sum_{m=-\infty}^{\infty}\frac{2(k+2m)}{p}J_{k+2m}(p)J_m(q). \tag{B6}$$

From Eqs. (B1) and (B4) one obtains Eq. (17). The parameters p,q, and ν are given in Eq. (14).

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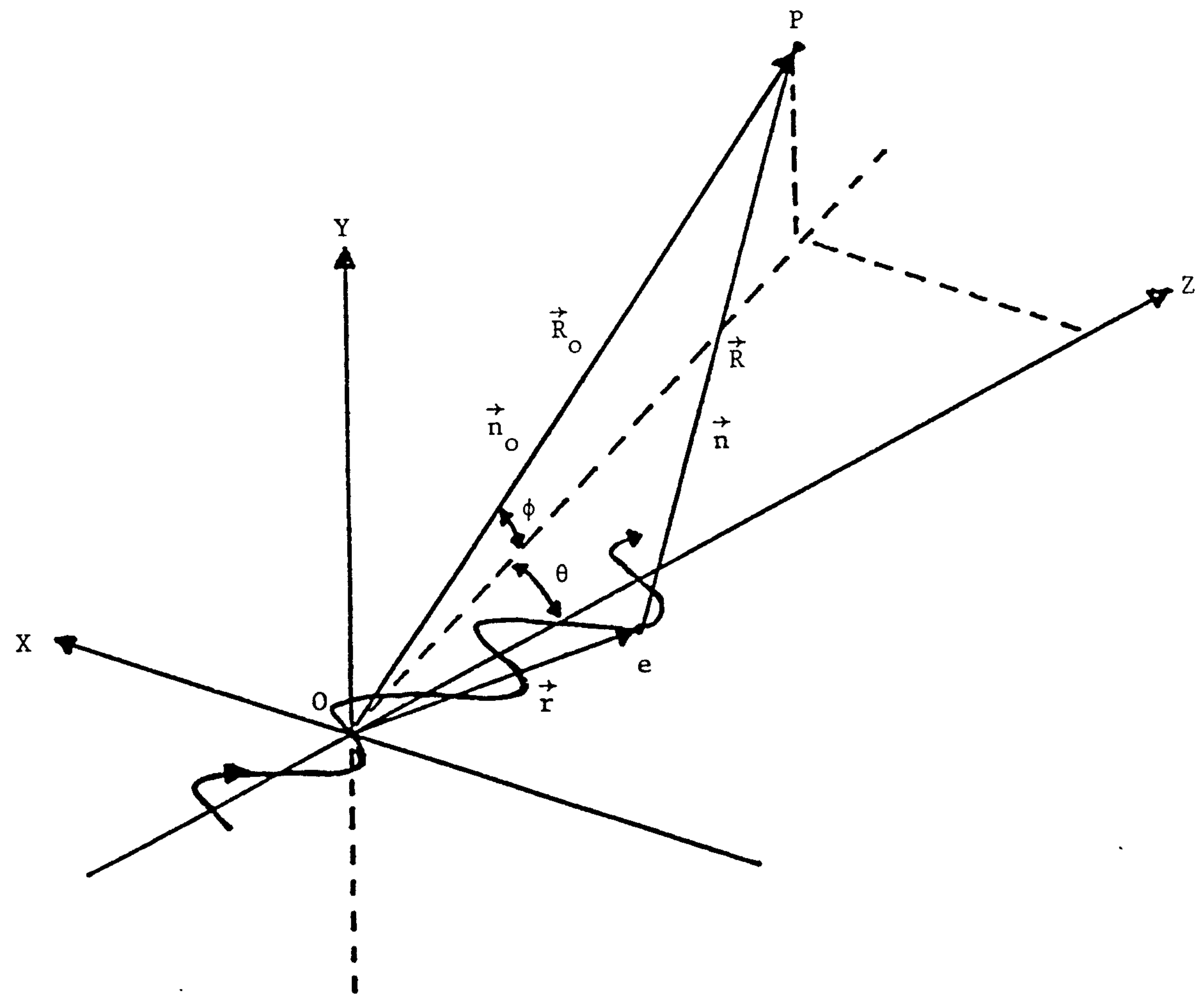

Fig. 1. The angle θ is in the XZ-plane and angle ϕ is in the YZ-plane of the coordinate system. The unit vectors $\vec{n}_o$ and $\vec{n}$ are from the origin O and from the election e to P, respectively.

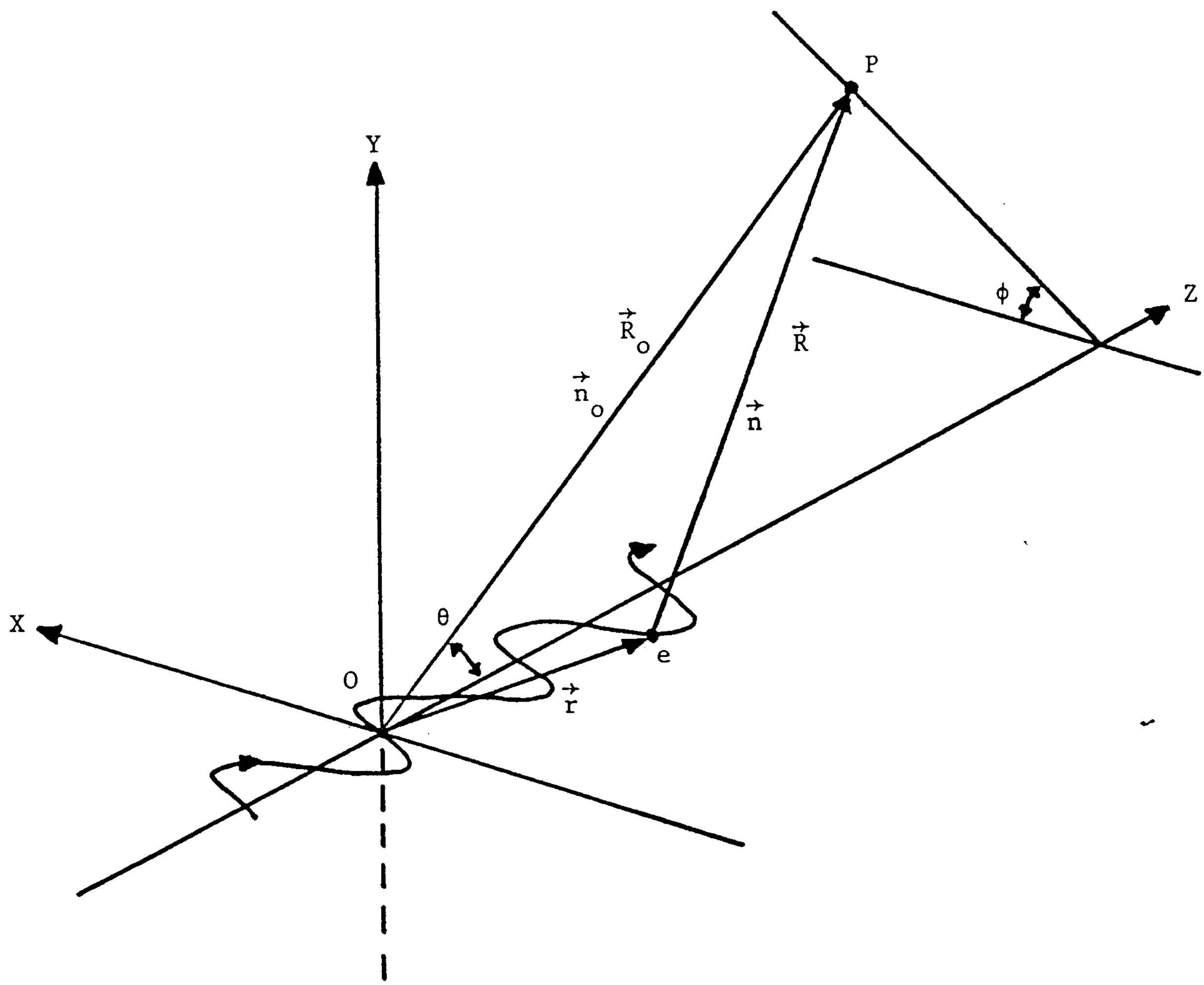

Fig. 2. Spherical coordinate system.